2 Covariance and Correlation — 4.2

2.1 Expectation of a function of two random variables

Expectation of a function of two random variables

Univariate: If X is a rv with pmf or pdf f and h is a function from reals to real, then

$$E(h(X)) = \begin{cases} \sum h(x)f(x), & \text{X is discrete} \\ \int_{-\infty}^{\infty} h(x)f(x)\,dx, & \text{X is continuous} \end{cases}$$

The sum for X discrete is over the possible values of X — those that have positive probability — called the range X. Bivariate: If X and Y have pmf or pdf f and h is a function from the plane to the plane, then

$$E(h(X,Y)) = \begin{cases} \sum h(x,y)f(x,y), & \text{X,Y are discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)f(x,y)\,dxdy, & \text{X,Y are continuous} \end{cases} \tag{5}$$

Similarly, for (X,Y) discrete, the sum is over the *pairs* (x,y) with positive probability — the range of X,Y.

Expectation of a function of two random variables

Textbook calls the range in both cases the *support* of the random variable.

Illustration with linear combination of two random variables

If $h(x,y) = ax + by, a, b \in \mathbb{R}$ then, using equation (5),

$$\begin{aligned} E(aX+bY) &= E(h(X,Y)) \\ &= \begin{cases} \sum (ax+by)f(x,y), & \text{X,Y discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (ax+by)f(x,y)\,dxdy, & \text{X,Y continuous} \end{cases} \\ &= \begin{cases} a\sum xf(x,y) + b\sum yf(x,y) & \text{X,Y discrete} \\ a\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)\,dxdy + b\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)\,dxdy, & \text{X,Y continuous} \end{cases} \\ &= \begin{cases} a\sum xf_X(x) + b\sum yf_Y(y), & \text{X,Y discrete} \\ a\int_{-\infty}^{\infty} xf_X(x)\,dx + b\int_{-\infty}^{\infty} yf_Y(y)\,dy, & \text{X,Y continuous} \end{cases} \\ &= aE(X) + bE(Y). \end{aligned}$$

This was demonstrated for discrete random variables using the sample space in Module 2 Section 2.6.

Variance of Sum

From Module 2 Section 4.1 , equation (26) showed that the variance of a sum of *any* random variables, X and Y can be written as

$$Var(X+Y) = Var(X) + Var(Y) + 2\,(E(XY) - E(X)E(Y))$$

Covariance is defined as the expression in brackets:

$$\begin{aligned} Cov(X,Y) &= E(XY) - E(X)E(Y) \\ &= E\left((X-\mu_X)(Y-\mu_Y)\right) \end{aligned} \tag{6}$$

following the argument in the next slide.